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AN INTRODUCTION TO LATTICES IN TWELVE-TONE MUSIC

Richard Kurth

University of British Columbia, School of Music
6361 Memorial Road,
Vancouver, B.C., CANADA V6T 1Z2
email: rkurth@unixg.ubc.ca

Twelve-tone music offers vital and diverse kinds of musical relationships and musical sense-making. These can nonetheless be appreciated without direct recourse to twelve-tone rows in particular. Every twelve-tone passage presents the set of all twelve tones as a collection of smaller discrete subsets, since the various tones group together in different ways. These collections of smaller discrete sets are called "partitions"; they represent equivalence relations on the collection of twelve tones, and can also be organized into lattices by defining the "meet" and "join" of any two partitions. The resulting "partition-lattices" can model multiplex musical relationships within a single passage, or musical continuity in a sequence of twelve-tone passages, by representing the interaction of various relevant partitions. Partition-lattices stimulate the exploration of listening strategies for twelve-tone music, and have significant implications for music theory and analysis. They are also of strictly mathematical interest, as lattices of equivalence relations.

Preliminaries

Every musical tone involves numerous defining characteristics: pitch (frequency), duration, loudness, timbre, and so forth. Pitch relations between musical tones are the principal organizing factor in all Western music and are the main focus of the following considerations; henceforth musical tones shall simply be referred to as "pitches," with other characteristics of individual tones being invoked only where relevant. In Western music, two pitches x and y are said to be an "octave" apart if their frequency ratio is 2:1. By the seventeenth century a 12-part division of the octave had become widespread, although a wide variety of tuning systems were used in practice (Barbour 1951, Blackwood 1985, Lindley and Turner-Smith 1993). Here we generally assume "equal temperament," in which the 2:1 octave is divided into 12 equal "semitones," each corresponding to the "interval" or frequency ratio $2^{1/12}$.

The equivalence relation most basic to twelve-tone music, and indeed to all Western music, is "octave equivalence": two pitches x and y are said to be members of the same *pitch class* if their frequency ratio is 2^n, where n is an integer (positive, negative, or 0). In 12-part equal temperament, the 12 pitch classes (henceforth pcs) are isomorphic to the integers modulo 12. We adopt the following conventional relation between traditional pc letternames used by musicians and the integers modulo 12 (written in hexadecimal notation): C = 0, C# = 1, D = 2, D# = 3, E = 4, F = 5, F# = 6, G = 7, G# = 8, A = 9, A# = A, B = B. (Given equal temperament, we also assume "enharmonic equivalence": C# = Db, D# = Eb, F# = Gb, G# = Ab, A# = Bb, etc.) The *unordered* set of all 12 pcs, {0,1,2,3,4,5,6,7,8,9,A,B}, is called the pc *aggregate*.

In twelve-tone compositions, a *twelve-tone row* (a strict serial ordering of the pc aggregate) is used to organize pc aggregates in various ways. (Given a 12-pc aggregate, there are 12! distinct twelve-tone rows.) Typically the pitch structures in such compositions are derived from a single row that is subjected to various permutations, the most common being the 12 *transpositions* (Tn) and the 12 *inversions* (In), defined as follows: for any pc x, and for n = 0, 1, ... , 11, $\mathrm{Tn}(x) = (x+\mathrm{n})\mathrm{mod}12$ and $\mathrm{In}(x) = (\mathrm{n}-x)\mathrm{mod}12$. The group structure of the 12 transposition and 12 inversion operations is well-known to music theorists (Starr 1978, Lewin 1987, Morris 1987). Other permutations are also used by some composers, but they shall not be addressed directly here.

Although twelve-tone rows are strictly ordered in the abstract, they are often presented in musical textures as partially ordered sets. For instance, an ordered *segment* from a row may be represented by simultaneous pitches; likewise, two (or more) ordered segments from a row may be presented concurrently (in "counterpoint") by independent voices or instruments. Such compositional practices, and others like them, can make accurate perception of twelve-tone rows difficult for listeners. The accurate perception of rows is not, in any case, the sole aesthetic goal of twelve-tone music; moreover, the theoretical and analytical problems posed by such compositional practices can be modelled by another sort of equivalence relation.

A (pitch-class) *partition* is an (unordered) collection of discrete (unordered) subsets of the pc aggregate, called the *membersets* of the partition, such that every pc is an element of one and only one memberset. Here partitions are labelled with boldface lowercase letters; membersets are separated by vertical slashes (|), and their elements written (by convention only) in numerical order, e.g. **v** = |015|234|6AB|789|. That notation is simply a more compact way of writing, more properly, **v** = { {0,1,5}, {2,3,4}, {6,A,B}, {7,8,9} }.

In principle, any partition, such as **v**, could model an infinite number of different twelve-tone textures, which might in turn be derived, in principal, from *any* twelve-tone row. To model a twelve-tone texture, the pcs in each of the four (unordered) membersets of **v** might all share the same value in some relevant musical parameter: for instance, the pcs in the (unordered) memberset {0,1,5} might all have the same attack point, or might be played by the same instrument, or have the same duration value, dynamic level, etc., regardless of other differences, and regardless of ordering in any other dimension or parameter.

```
             (a)              (b)                 (c)                (d)
Violin 1:    | 0  2  6        | 051               | B A         6    |        789   |
Violin 2:    | 1  3     7     |    6AB            |  0        1 5    |         324  |
Viola:       | 5     A  8     |       432         |    9  78         |  AB6         |
Cello:       |    4  B  9     |          789      |     24  3        | 510          |
```

Figure 1. Four aggregates projecting v in different ways.

By way of example, the abstract schema in Figure 1 represents a (four-voice) string quartet texture, and shows four different aggregates (separated by vertical slashes), each projecting **v** in a different way. Temporal order is represented on the figure by spacing left-to-right, and simultaneities are vertically aligned; horizontal strata represent instrumental assignment. Aggregate (a) presents the membersets of **v** as simultaneities, each one involving a different selection of three instruments among the four in the quartet; each simultaneity could be ordered in 3! ways, consequently, aggregate (a) could be derived from $(3!)^4 = 1{,}296$ different twelve-tone rows. Aggregate (b) presents each **v** memberset in a strict order, and in a single instrument; this aggregate can be derived from only 1 twelve-tone row, but it is not one of rows relevant to aggregate (a). Like (b), aggregate (c) also presents each **v** memberset in a strict order and single instrument; (c) can be derived from only 1 twelve-tone row, distinct from all those relevant to (a) or (b). Aggregate (d) also presents each **v** memberset in strict order and single instrument; but because of its four dyadic simultaneities ({1,A}, {0,B}, {3,8}, and {2,9}), it can be derived from $(2!)^4 = 16$ different twelve-tone rows, all distinct from those relevant to the preceding aggregates.

The *format* of a partition **p**, denoted FORMAT(**p**), simply lists the number of elements in each of its membersets. For **v** as defined above, FORMAT(**v**) = [3333]. For an aggregate of 12 pcs (regardless of temperament), there are 77 different partition formats (see the Appendix for a listing). Twelve-tone music theorists (Starr 1978, Mead

1988, Morris and Alegant 1988, Alegant 1993, Kurth 1993 and 1996) have examined some formal properties of pc partitions, but partitions have mostly been used "practically," to model and analyze musical textures. The one-to-one relation between partitions and equivalence relations on the aggregate has gone unnoted and unexplored in the music-theoretical literature, largely because music analysts are uncomfortable suggesting that two (or more) different pcs are "equivalent." Nonetheless, it is easily shown that there are precisely 4,213,597 distinct (twelve-tone) pc partitions. Given the one-to-one correspondence between partitions and equivalence relations, there will be an equal number of equivalence relations on any set of 12 elements. A formula and a table enumerating the number of distinct partitions for each of the 77 formats is provided in the Appendix.

A partition **p** is a *subpartition* of a partition **q** (henceforth $\mathbf{p} \subset \mathbf{q}$) if every memberset of **p** is a subset of some memberset of **q**. Any collection of partitions can be partially ordered under this inclusion relation. Partitions can also be organized into lattices, as will be seen shortly.

Two partition formats have unique representatives which will be of use later on: the "conjunct" partition, **conj** = |0123456789AB|, and the "disjunct" partition, **disj** = |0|1|2|3|4|5|6|7|8|9|A|B|. **conj** is the "greatest" partition of the aggregate, and every partition is a subpartition of **conj**; likewise, **disj** is the "least" partition of the aggregate, and **disj** is a subpartition of every partition.

Partition meet and partition join

In order to construct lattices from partitions later on, we define the meet and join of any two partitions **p** and **q**. We shall not prove here that partition meet and join as defined, are partitions, but that fact will be corroborated by selected examples.

Definition (*partition meet*). Let **p** and **q** be (pc) partitions, let x and y be any two pcs. Then x and y are in the same memberset of $\mathbf{p} \wedge \mathbf{q}$, the *meet* of **p** and **q**, if and only if x and y are in the same memberset of **p** and also in the same memberset of **q**. Equally, let P and Q be the equivalence relations represented by **p** and **q** respectively. Then the partition meet $\mathbf{p} \wedge \mathbf{q}$ corresponds with the equivalence relation M, where $x\mathrm{M}y$ if and only if $x\mathrm{P}y$ and $x\mathrm{Q}y$.

Definition (*partition join*). Let **p** and **q** be (pc) partitions, let x and y be any two pcs. Then x and y are in the same memberset of $\mathbf{p} \vee \mathbf{q}$, the *join* of **p** and **q**, if x and y are in the same memberset of **p** or if x and y are in the same memberset of **q**. Equally, let P and Q be the equivalence relations represented by **p** and **q** respectively. Then the partition join $\mathbf{p} \vee \mathbf{q}$ corresponds with the equivalence relation J, where $x\mathrm{J}y$ if $x\mathrm{P}y$ or if $x\mathrm{Q}y$.

Example. Let **v** = |015|234|6AB|789|, as above, and let **w** = |045|123|67B|89A|. Then $\mathbf{v} \wedge \mathbf{w}$ = |05|1|23|4|6B|7|89|A| and $\mathbf{v} \vee \mathbf{w}$ = |012345|6789AB|.

The example illustrates that for any two partitions, **p** and **q**, $\mathbf{p} \wedge \mathbf{q} \subset \mathbf{p} \subset \mathbf{p} \vee \mathbf{q}$, and $\mathbf{p} \wedge \mathbf{q} \subset \mathbf{q} \subset \mathbf{p} \vee \mathbf{q}$.

As defined, partition meet and join are commutative and also associative, although those facts shall not be proven here. The following lemmas are also given without proof.

Lemma. The following three conditions are equivalent: (1) $\mathbf{p} \subset \mathbf{q}$; (2) $\mathbf{p} \wedge \mathbf{q} = \mathbf{p}$; (3) $\mathbf{p} \vee \mathbf{q} = \mathbf{q}$.

Lemma. The join $\mathbf{p} \vee \mathbf{q}$ is the *least upper bound* of {**p**, **q**}. That is, if **r** is a superpartition of both **p** and **q**, then $\mathbf{p} \vee \mathbf{q} \subset \mathbf{r}$. Similarly, the meet $\mathbf{p} \wedge \mathbf{q}$ is the *greatest lower bound* of {**p**, **q**}. That is, if **s** is a subpartition of both **p** and **q**, then $\mathbf{s} \subset \mathbf{p} \wedge \mathbf{q}$.

If $\mathbf{p} \wedge \mathbf{q}$ = **disj**, we shall say that **p** and **q** are *disjoint*. If $\mathbf{p} \vee \mathbf{q}$ = **conj**, we shall say that **p** and **q** are *conjoint*. If **p** and **q** are disjoint and conjoint, we shall say that they are *conjugates*. Every partition has at least one disjoint partition (**disj**), and at least one conjoint partition (**conj**). For any partition **p**, the set of its conjugate partitions is the intersection of the set of its disjoint partitions with the set of its conjoint partitions. Only **conj** and **disj** have unique conjugate partitions, **disj** and **conj** respectively.

Partition lattices

Without developing all the necessary mathematical formalities here, we simply offer the following definition.

Definition (*partition lattice*). Let **L** = {**a**, **b**, **c**, ... } be a collection of (pc) partitions. **L** is a *partition lattice* if for every **p** and **q** $\in$ **L**, the meet **p** $\wedge$ **q** $\in$ **L** and the join **p** $\vee$ **q** $\in$ **L**.

Example. Figure 2 shows the (four-part) pre-compositional "array" for the first 12 measures of Milton Babbitt's *Semi-Simple Variations* (1956), a short composition for piano solo. A musical score for the passage will be shown in the conference presentation, and a number of observations made at that time about different listening strategies relevant to the music. Figure 2 indicates, for the present, that the passage involves four voices, or strata of relative pitch height, here labelled "Soprano" (highest), "Alto" (mid-high), "Tenor" (mid-low), and "Bass" (lowest). The array abstracts properties of the actual music, and vertical alignment *does not* represent simultaneity in this case. But the array shows how each voice presents a different strict ordering of the 12 pcs, as four consecutive ordered trichords, so that the entire texture presents four different partitions: **r** = |045|123|6AB|789| and **s** = |015|234|67B|89A| are each presented "horizontally," in two different ways, by aggregates in two individual voices; meanwhile **v** and **w** (already encountered in earlier examples), are each presented twice by aggregates generated "column-wise" by all four voices together. All four partitions have the same format, [3333]. Readers may confirm for themselves that **r** and **s** map onto one another under I5 and I11 (as defined above), and are each invariant (map onto themselves) under T0 and T6; similarly, **v** and **w** map onto one another under I5 and T6, and invariant under T0 and I11.

```
(Soprano)    | <A,6,B> | <8,7,9> | <3,1,2> | <5,0,4> |      r
(Alto)       | <2,4,3> | <0,5,1> | <7,B,6> | <9,A,8> |      s
(Tenor)      | <9,7,8> | <B,6,A> | <4,0,5> | <2,1,3> |      r
(Bass)       | <1,5,0> | <3,4,2> | <8,A,9> | <6,B,7> |      s
                  v         v         w         w
```

Figure 2. Four-part array for Milton Babbitt's "Semi-Simple Variations," *mm. 1-12.*

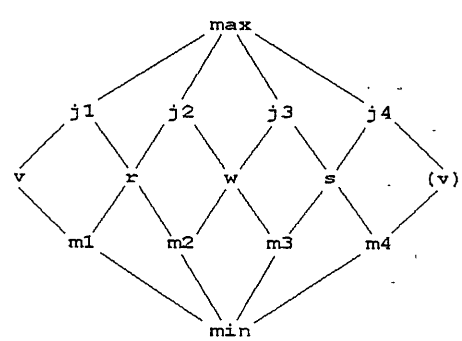

Figure 3. Partition lattice **L1** *generated from* **r**, **s**, **v**, *and* **w**.

Figure 3 offers a two-dimensional representation of the 14-element partition lattice **L1** that can be generated from the four array partitions, **r**, **s**, **v**, and **w**. The joins **ji** and meets **mi** (for i = 1, 2, 3, 4) are not written out in detail here, but the greatest and least partitions, labelled **max** and **min** on the figure, were already encountered in Example 1: **max** = **v** $\vee$ **w** = |012345|6789AB| = **r** $\vee$ **s**, and **min** = **v** $\wedge$ **w** = |05|1|23|4|6B|7|89|A| = **r** $\wedge$ **s**. In line with the transformational properties just observed above, the twelve-tone operations T0, T6, I5, and I11 are automorphisms on **L1**. The cyclic permutations $\sigma 1 = (14)$, $\sigma 2 = (7A)$, and $\sigma 3 = (14)(7A)$ are also automorphisms on **L1**, since $\sigma 1(\mathbf{r}) = \mathbf{v}$, $\sigma 2(\mathbf{r}) = \mathbf{w}$, and $\sigma 3(\mathbf{r}) = \mathbf{s}$, etc.

Partition lattices and listening strategies

The first 12 measures of Babbitt's *Semi-Simple Variations* can be explored further, to illustrate the usefulness of partition lattices for musical analysis. First we note that the array partitions **r**, **s**, **v**, and **w** are all very relevant to a listening experience of the passage. (The conference presentation will demonstrate, using the musical score and an audio recording.) The partitions **max** and **min** are much less relevant, however, and the joins **ji** and meets **mi** do not provide much musical insight into the passage. Here we shall explore some "ad hoc" partitions that offer compelling alternative listening strategies for sections of the music. These partitions are at some odds with the four partitions of the pre-compositional array, and what follows shows that partition lattices can be used to explore how "ad hoc" partitions and listening strategies interact with the pre-compositional array partitions.

The "dyadic" partition **d** = |03|19|2A|46|57|8B|, with format [222222], offers a rewarding listening strategy for the first 6 measures (as will be explored and demonstrated in the conference presentation). It also has the property of being a conjugate for all of the array partitions, **r**, **s**, **v**, and **w**. That fact is of musical interest. On the one hand **d** "integrates" the entire texture, in the sense that its join with each of the array partitions is **conj**; and on the other hand **d** also "decomposes" the entire texture, in the sense that its meet with each of the array partitions is **disj**. As a result, **d** can be said to "completely synthesize" and also to "completely analyze" the texture in mm. 1-6. The conjugate properties of **d** are also of theoretical interest, since they lead to four small but relevant partition lattices: {**d**, **r**, **conj**, **disj**}, {**d**, **s**, **conj**, **disj**}, {**d**, **v**, **conj**, **disj**}, and {**d**, **w**, **conj**, **disj**}.

conj =	|0123456789AB|
c =	|02357A|14689B|
x =	|0235|14|689B|7A|
d =	|03|19|2A|46|57|8B|
r =	|045|123|6AB|789|
s =	|015|234|67B|89A|
y =	|0235|1|4|689B|7|A|
min =	|05|1|23|4|6B|7|89|A|
e =	|03|1|2|4|5|6|7|9|8B|A|
disj =	|0|1|2|3|4|5|6|7|8|9|A|B|

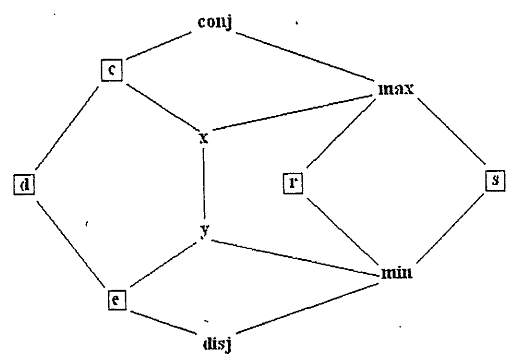

Figure 4. Partition lattice **L2** *generated from* **d**, **r**, *and* **s**.

Figure 4 explores how **d** interacts only with the 2 "horizontal" array partitions, **r** and **s**, and represents the resulting partition lattice, **L2**. The specific contents of all 11 partitions are listed on the figure. Aside from the array partitions **r** and **s**, the partitions **c**, **d**, and **e** are more relevant to the listening experience (in ways to be demonstrated at the conference presentation) than any other partitions in the lattice; the labels for these five partitions are enclosed in boxes for visual emphasis. The graphic representation of lattice **L2** helps show how **c**, **d**, **e**, are all at some odds with **r** and **s**. In particular, $\mathbf{c} = \mathbf{d} \vee \mathbf{min}$ and $\mathbf{e} = \mathbf{d} \wedge \mathbf{max}$ result from the join and meet interaction of **d** with $\mathbf{max} = \mathbf{r} \vee \mathbf{s}$ and $\mathbf{min} = \mathbf{r} \wedge \mathbf{s}$, so **c** and **e** mediate between **d** and the interaction of its two conjugates, **r** and **s**. The inversion operation I11 is an automorphism on **L2**; **r** and **s** map to one another under I11, while every other partition in the lattice is invariant under I11.

We can explore in a similar way how **d** interacts with the two "columnar" array partitions, **v** and **w**. In fact, the resulting lattice, **L3**, is otherwise identical to lattice **L2** except for substituting **v** and **w** in place of **r** and **s** on Figure 4. Once again, I11 is an automorphism on **L3**, because every partition in the lattice, including **v** and **w**, is invariant under I11. **L3** and **L2** are *isographic*, but they are *not* isomorphic; it appears to be impossible to find a permutation that will not only map **r** and **s** to **v** and **w** (the corresponding pairs of partitions of format [3333]) but that will also map **d** (the only partition with format [222222]) to itself.

Lattices **L2** and **L3** explored the interaction of **d** with the "horizontal" array partitions and "columnar" array partitions respectively, in order to examine in detail how **d**, as a listening strategy, engages the texture in mm. 1-6 of *Semi-Simple Variations*. Lattices can also be used to model musical continuities in the composition. For instance, while the dyadic partition **d** is a compelling listening strategy for mm. 1-6, the "trichordal" partition **t** = |048|126|37B|59A|, with format [3333], likewise reveals aspects of mm. 7-12, and the partition **z** = |0B|1234|56|789A|, with format [2244], is a useful listening strategy for mm. 13-18. Figure 5 shows the lattice **L4** generated from **d**, **t**, and **z**, which provides insight into how these three partitions and their corresponding listening strategies interact over the course of mm. 1-18. The musical relevance of the partitions in **L4**, including the meet **z'** and the join **d'** will be demonstrated in the conference presentation.

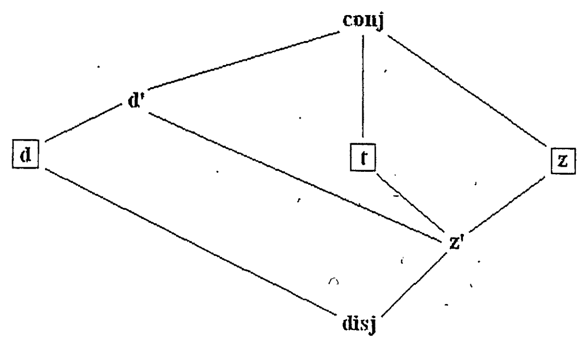

Figure 5. Partition lattice **L4** *generated from* **d**, **t**, *and* **z**.

Appendix

Enumeration. For any aggregate containing 12 distinct pcs, there are a total of 4,213,597 *distinct* pc partitions. Equivalently, there are 4,213,597 equivalence relations on any set containing 12 distinct elements. In particular, for each FORMAT = [*abc...mn*], the corresponding number of distinct partitions (or equivalence relations) is X/R, where X and R are calculated as follows:

$$X = \langle 12,a\rangle \times \langle 12\text{-}a,b\rangle \times \langle 12\text{-}a\text{-}b,c\rangle \times \ldots \times \langle 12\text{-}a\text{-}b\text{-}c\text{-}\ldots\text{-}m,n\rangle.$$

$$R = A! \times B! \times C! \times \ldots \times M! \times N!$$

In the formula for X, $\langle n,k\rangle$ is the binomial coefficient $n!/(k!(n-k)!)$. In the formula for R, A is the number of instances of memberset size a in the format, B is the number of instances of memberset size b, and so forth. The product X counts some partitions more than once, if any memberset size occurs more than once in the format. R offsets this problem by calculating the precise number of such counting redundancies. (This formula corrects the erroneous formula given in Alegant 1993.)

Table 1 calculates these values. Column **A** gives the number of distinct partitions for each format, and determines that their total sum is 4,213,597.

Format	A	B	Format	A	B
[C]	1	39,916,800	[11244]	103,950	3,742,200
			[12225]	83,160	1,995,840
[1B]	12	43,545,600	[11334]	138,600	3,326,400
[2A]	66	23,950,080	[12234]	415,800	4,989,600
[39]	220	17,740,800	[12333]	184,800	1,478,400
[48]	495	14,968,800	[22224]	51,975	311,850
[57]	792	13,685,760	[22233]	138,600	554,400
[66]	462	6,652,800			
			[111117]	792	570,240
[11A]	66	23,950,080	[111126]	13,860	1,663,200
[129]	660	26,611,200	[111135]	27,720	1,330,560
[138]	1,980	19,958,400	[111144]	17,325	623,700
[228]	1,485	7,484,400	[111225]	83,160	1,995,840
[147]	3,960	17,107,200	[111234]	277,200	3,326,400
[156]	5,544	15,966,720	[111333]	61,600	492,800
[237]	7,920	11,404,800	[112224]	207,900	1,247,400
[246]	13,860	9,979,200	[112233]	415,800	1,663,200
[255]	8,316	4,790,016	[122223]	207,900	415,800
[336]	9,240	4,435,200	[222222]	10,395	10,395
[345]	27,720	7,983,360			
[444]	5,775	1,247,400	[1111116]	924	110,880
			[1111125]	16,632	399,168
[1119]	220	8,870,400	[1111134]	27,720	332,640
[1128]	2,970	14,968,800	[1111224]	103,950	623,700
[1137]	7,920	11,404,800	[1111233]	138,600	554,400
[1227]	11,880	8,553,600	[1112223]	277,200	554,400
[1146]	13,860	9,979,200	[1122222]	62,370	62,370
[1155]	8,316	4,790,016			
[1236]	55,440	13,305,600	[11111115]	792	19,008
[2226]	13,860	1,663,200	[11111124]	13,860	83,160
[1245]	83,160	11,975,040	[11111133]	9,240	36,960
[1335]	55,440	5,322,240	[11111223]	83,160	166,320
[1344]	69,300	4,989,600	[11112222]	51,975	51,975
[2235]	83,160	3,991,680			
[2244]	51,975	1,871,100	[111111114]	495	2,970
[2334]	138,600	3,326,400	[111111123]	7,920	15,840
[3333]	15,400	246,400	[111111222]	13,860	13,860
[11118]	495	2,494,800	[1111111113]	220	440
[11127]	7,920	5,702,400	[1111111122]	1,485	1,485
[11136]	18,480	4,435,200			
[11226]	41,580	4,989,600	[11111111112]	66	66
[11145]	27,720	3,991,680			
[11235]	166,320	7,983,360	[111111111111]	1	1
...continued...	...continued..	...continued...	Totals	**4,213,597**	**479,001,600 = 12!**

Table 1. Enumerating the distinct partitions for each of the 77 formats.

Column **B** on Table 1 allows us to confirm the numbers generated in column **A**, by determining the number of *permutations* with cycle-lengths corresponding to the partition formats. Each partition memberset is an unordered set; but its contents could be ordered as a permutation cycle in (j-1)! different ways, where j is the size of the memberset. Consequently, for FORMAT = [*abc ... mn*] there are:

$$X/R \times (a\text{-}1)! \times (b\text{-}1)! \times (c\text{-}1)! \times \ldots (m\text{-}1)! \times (n\text{-}1)!$$

different permutations with the corresponding cycle lengths. Column **C** calculates these values, and determines that their sum over all 77 formats is 479,001,600 = 12! This is the total number of possible permutations of 12 elements, so it confirms the X/R values in column **A**.

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